

CANONICAL DECOMPOSITION OF THE COHOMOLOGY GROUPS OF NILPOTENT LIE ALGEBRAS

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ABSTRACT. Through a natural filtration of the Chevalley-Eilenberg differential complex for real nilpotent Lie algebras a spectral sequence that converge to the Lie algebra cohomology arises. From this sequence we define what we call intermediate cohomology; we give examples and study its properties. Finally we compute it for all nilpotent Lie algebras up to dimension six.

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1. INTRODUCTION

After Nomizu's work [13], the cohomology of a nilpotent Lie algebra \mathfrak{g} is interpreted as the de Rham cohomology of a nilmanifold, which are homogeneous manifolds of the form $M = \Gamma \backslash G$ where G is a connected simply-connected nilpotent Lie group and Γ a compact discrete subgroup.

In this sense, invariant forms on a nilmanifold M are elements of the exterior algebra of \mathfrak{g} . Moreover, the existence of certain geometric structures on M that are invariant by the action of the group G (symplectic, complex, among others) has implications on the Lie algebra cohomology (see for instance [15, 4, 3]). Actually, nilmanifolds admit no Kähler structures because of cohomological reasons (see [1] or [14]).

The computation of the cohomology of nilpotent Lie algebras is not easy and it is usually studied separately by families (free, filiform, etc...) using different approaches [2, 18, 5]. A standing conjecture in this subject is the Toral Rank Conjecture due to S. Halperin [6], which asserts that the total dimension of the Lie algebra cohomology ring of \mathfrak{g} is greater than the total dimension of the Lie algebra cohomology ring of the center of \mathfrak{g} .

From this point of view, a better understanding of the cohomology of nilpotent Lie algebras is required.

The purpose of this work is to develop a new concept in Lie algebra cohomology of nilpotent Lie algebras called the *intermediate cohomology* of the Lie algebra; it was first introduced by Simon Salamon (in personal communication with Isabel Dotti) and our goal is to go deeper on its properties.

Given a nilpotent Lie algebra \mathfrak{g} over the field of real numbers, the intermediate cohomology groups of degree i are a finite number of vector spaces $E_{\infty}^{p,q}$ that satisfy

$$H^i(\mathfrak{g}) \cong \bigoplus_{p+q=i} E_{\infty}^{p,q},$$

where $H^i(\mathfrak{g})$ is the i -th cohomology group of \mathfrak{g} with trivial coefficients. This important property is the reason why we name these groups the intermediate cohomology groups of \mathfrak{g} .

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To construct those groups we deal with a natural filtration of the Chevalley-Eilenberg differential complex of the nilpotent Lie algebra, namely, the annihilator spaces of the central descending series. This filtration induces a cohomological spectral sequence that converges to the Lie algebra cohomology with trivial coefficients, obtaining the previous formula.

Our main contribution is, on the one hand, the study of properties of this new concept and, on the other hand, the computation of the intermediate cohomology (and hence the Lie algebra cohomology) of all nilpotent Lie algebras up to dimension six. Nilpotent Lie algebras over \mathbb{R} are classified up to dimension seven [10]. Though, dimension six is the highest dimension in which there do not exist continuous families ([10, 8, 16, 12]).

An outline of the paper is as follows. In Section 2 we review all necessary definitions concerning Lie algebra cohomology, the filtration of the Chevalley-Eilenberg complex and facts regarding the spectral sequence that arises from that filtration. Throughout this work elementary knowledge of spectral sequences will be assumed. In Section 3 we present the formal definition of intermediate cohomology, examples and general properties. We introduce a way to display the intermediate cohomology of a Lie algebra in Section 4. Section 5 is devoted to the computation of the intermediate cohomology of all nilpotent Lie algebras up to dimension six.

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2. LIE ALGEBRA COHOMOLOGY AND THE SPECTRAL SEQUENCES OF A NILPOTENT LIE ALGEBRA

Let \mathfrak{g} denote a real Lie algebra. The central descending series of \mathfrak{g} , $\{\mathfrak{g}^i\}$ for all $i \geq 0$, is given by

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}], \quad i \geq 1.$$

A Lie algebra \mathfrak{g} is k -step nilpotent if $\mathfrak{g}^k = 0$ and $\mathfrak{g}^{k-1} \neq 0$; this number k is called the nilpotency index of \mathfrak{g} . For example, abelian Lie algebras are one step nilpotent. Moreover, 2-step nilpotent Lie algebra verify $\mathfrak{g}^1 \subseteq \mathfrak{z}(\mathfrak{g})$, where $\mathfrak{z}(\mathfrak{g})$ denotes the center of \mathfrak{g} .

Let us consider the Chevalley-Eilenberg complex of \mathfrak{g}

$$(1) \quad \mathbb{R} \xrightarrow{d=0} \mathfrak{g}^* = C^1(\mathfrak{g}) \xrightarrow{d} C^2(\mathfrak{g}) \xrightarrow{d} \dots \xrightarrow{d} C^p(\mathfrak{g}) \xrightarrow{d} \dots$$

Here $C^p(\mathfrak{g})$ denotes the vector space of skew-symmetric p -linear forms on \mathfrak{g} identified with the exterior product $\Lambda^p \mathfrak{g}^*$, and the differential $d : \Lambda^p \mathfrak{g}^* \rightarrow \Lambda^{p+1} \mathfrak{g}^*$ is defined by:

$$d_p c(x_1, \dots, x_{p+1}) = \sum_{1 \leq i < j \leq p+1} (-1)^{i+j-1} c([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{p+1}).$$

Notice that the differential $d : \mathfrak{g} \rightarrow \Lambda^2 \mathfrak{g}^*$ coincides with the dual mapping of the Lie bracket $[\cdot, \cdot] : \Lambda^2 \mathfrak{g} \rightarrow \mathfrak{g}$ and for $\rho, \eta \in \Lambda^*(\mathfrak{g}^*)$ we have

$$d(\rho \wedge \eta) = d\rho \wedge \eta + (-1)^{\deg \rho} \rho \wedge d\eta.$$

Definition 2.1. The cohomology of $(C^*(\mathfrak{g}), d)$ is called the Lie algebra cohomology (with trivial coefficients) of \mathfrak{g} and it is denoted by $H^*(\mathfrak{g})$.

In what follows we introduce subspaces of the dual of a nilpotent Lie algebra considered in [16] which lead us to a canonical filtration of the complex in (1).

Define the following subspaces of \mathfrak{g}^*

$$(2) \quad V_0 = \{0\} \quad V_i = \{\alpha \in \mathfrak{g}^* : d\alpha \in \Lambda^2 V_{i-1}\} \quad i \geq 1.$$

Notice that V_1 is the space of closed 1-forms and that $V_0 \subseteq V_1 \subseteq \dots \subseteq V_i \subseteq \dots \subseteq \mathfrak{g}^*$. Moreover, in [16] is proved that these spaces are dual to those in the central descending series. That is, for each $i \geq 0$, $V_i = \{x^* \in \mathfrak{g}^* : x^*(u) = 0, \forall u \in \mathfrak{g}^i\} = (\mathfrak{g}^i)^\circ$. In particular, \mathfrak{g} is a k -step nilpotent Lie algebra if and only if $V_k = \mathfrak{g}^*$ and $V_{k-1} \neq \mathfrak{g}^*$.

Let \mathfrak{g} be a k -step nilpotent Lie algebra of dimension m . The sequence in (2) defines a filtration in the space of skew symmetric q -forms, $\Lambda^q \mathfrak{g}^*$, for all $q = 0, \dots, m$,

$$(3) \quad 0 = \Lambda^q V_0 \subsetneq \Lambda^q V_1 \subsetneq \dots \subsetneq \Lambda^q V_{k-1} \subsetneq \Lambda^q V_k = \Lambda^q \mathfrak{g}^*.$$

Also $d(\Lambda^q V_i) \subset \Lambda^{q+1} V_i$ so each of these subspaces is invariant under the differential. Furthermore, fixed p ,

$$(4) \quad F^p C^* : 0 \longrightarrow \mathbb{R} \longrightarrow V_{k-p} \longrightarrow \Lambda^2 V_{k-p} \longrightarrow \dots \longrightarrow \Lambda^m V_{k-p} \longrightarrow 0$$

is a subcomplex of the Chevalley-Eilenberg complex. Clearly $0 \equiv F^k C^* \subseteq F^{k-1} C^* \subseteq \dots \subseteq F^{p+1} C^* \subseteq F^p C^* \subseteq \dots \subseteq F^1 C^* \subseteq F^0 C^* = C^*$. This implies that $\{F^p C^*\}$ constitutes a filtration of the complex (1).

This filtration gives rise to a spectral sequence $\{E_r^{p,q}\}_{r \geq 0}^{p,q \in \mathbb{Z}}$ where

$$(5) \quad E_0^{p,q} = \frac{F^p C^{p+q}}{F^{p+1} C^{p+q}}, \quad E_1^{p,q} = H^{p+q}(F^p C^* / F^{p+1} C^*), \quad E_\infty^{p,q} = \frac{F^p H^{p+q}(C^*)}{F^{p+1} H^{p+q}(C^*)}.$$

For any Lie algebra \mathfrak{g} , the spectral sequence that arises in this natural way is always bounded because the dimension of \mathfrak{g} and its nilpotency index are finite. It is well known that given a cohomological complex and a filtration of it, the associated spectral sequence converges to the cohomology of the original complex. For this fact and further details on homological algebra, we refer the reader to [7, 19].

Therefore, the spectral sequence in (5) converges to the Lie algebra cohomology of \mathfrak{g} . This fact is denoted as

$$E_r^{p,q} \Rightarrow H^*(\mathfrak{g}).$$

An isomorphism between two nilpotent Lie algebras preserves the filtrations and induces a spectral sequence isomorphism. This implies that there is a spectral sequence associated to each isomorphism class of nilpotent Lie algebras.

Properties of spectral sequences arising from a filtration of a cohomology complex apply to this particular one, yielding the following result.

Proposition 2.2. *Let \mathfrak{g} be a nilpotent Lie algebra and consider the natural filtration given in (4). Then the spectral sequence (5) satisfies*

$$(1) \quad E_0^{p,q} = 0 \text{ if } p < 0 \text{ or } p \geq k. \text{ Moreover for } 0 \leq p \leq k-1$$

$$(6) \quad E_0^{p,q} = \frac{F^p C^{p+q}}{F^{p+1} C^{p+q}} = \frac{\Lambda^{p+q} V_{k-p}}{\Lambda^{p+q} V_{k-(p+1)}}.$$

$$(2) \quad \text{For each } n \in \mathbb{N}_0 \text{ and } r \geq 0, \text{ the elements of total degree } n \text{ are } E_r^{n,0}, E_r^{1,n-1}, \dots, E_r^{k,n-k} \text{ where } k \text{ is the nilpotency index of } \mathfrak{g}.$$

(3) In particular, for $r = 0$ the elements of degree n are the quotient spaces:

$$\frac{\Lambda^n \mathfrak{g}^*}{\Lambda^n V_{k-1}}, \frac{\Lambda^n V_{k-1}}{\Lambda^n V_{k-2}}, \dots, \frac{\Lambda^n V_1}{\Lambda^n V_0}, \Lambda^n V_0.$$

Each term of the spectral sequence can be computed directly by calculating d paying attention to the filtration. Explicitly,

$$(7) \quad E_r^{p,q} = \frac{\{x \in \Lambda^{p+q} V_{k-p} : dx \in \Lambda^{p+q+1} V_{k-p-r}\}}{d(\{x \in \Lambda^{p+q-1} V_{k-p+r-1} : dx \in \Lambda^{p+q} V_{k-p}\}) + \{x \in \Lambda^{p+q} V_{k-p-1} : dx \in \Lambda^{p+q+1} V_{k-p-r}\}},$$

and the limit term is

$$(8) \quad E_\infty^{p,q} = \frac{\{x \in \Lambda^{p+q} V_{k-p} : dx = 0\}}{d(\{x \in \Lambda^{p+q-1} \mathfrak{g}^* : dx \in \Lambda^{p+q} V_{k-p}\}) + \{x \in \Lambda^{p+q} V_{k-p-1} : dx = 0\}}.$$

3. INTERMEDIATE COHOMOLOGY: PROPERTIES AND EXAMPLES

In the previous section, to each nilpotent Lie algebra \mathfrak{g} was associated a natural filtration of its dual space and a filtration of the Chevalley-Eilenberg differential complex. This filtration gives rise to a canonical spectral sequence that converges to the Lie algebra cohomology. This implies that each cohomology group $H^i(\mathfrak{g})$ can be written as a direct sum of the limit terms of the spectral sequence. Namely

$$(9) \quad H^i(\mathfrak{g}) \cong \bigoplus_{p+q=i} E_\infty^{p,q} \quad \text{for all } i = 0, \dots, m.$$

This way of describing the cohomology groups as a sum of smaller spaces suggests us the following definition.

Definition 3.1. Let \mathfrak{g} be a nilpotent Lie algebra of dimension m . Then, for each $i = 0, \dots, m$, the intermediate cohomology groups of degree i of \mathfrak{g} are the vector spaces $E_\infty^{p,q}$ with $p + q = i$.

3.1. Examples. To illustrate how to compute this cohomology and its relationship with the Lie algebra cohomology we deal with abelian Lie algebras and the nilpotent Lie algebra of dimension three.

Intermediate cohomology of abelian Lie algebras. Let \mathfrak{g} be the abelian Lie algebra of dimension m . Then $k = 1$ and therefore $V_1 = \mathfrak{g}^*$. The subcomplex $F^1 C^*$ is the zero complex and

$$F^0 C^* : 0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{g}^* \longrightarrow \Lambda^2 \mathfrak{g}^* \longrightarrow \dots \longrightarrow \Lambda^m \mathfrak{g}^* \longrightarrow 0.$$

From (1) in Proposition 2.2, $E_0^{p,q} = 0$ except for $p = 0$ and according to Formula (6)

$$E_0^{0,q} = \Lambda^{p+q} \mathfrak{g}^*.$$

Therefore, for $r = 0$ the terms coincide with those in the Chevalley-Eilenberg complex.

The differentials $d_0^{p,q} : E_0^{p,q} \longrightarrow E_0^{p,q+1}$ are all zero except when $p = 0$, and in that case they coincide with the Lie algebra differential $d : \mathfrak{g} \longrightarrow \Lambda^2 \mathfrak{g}^*$.

The E_1 term is the cohomology of the complexes in E_0 , then $E_1^{p,q} = \{0\}$ if $p \neq 0$. For $q = 1, \dots, m$, there is only one non zero term of degree q and it is $E_1^{0,q} = H^q(\mathfrak{g})$. This spectral sequence stabilizes in E_1 since $d_r^{p,q}$ has as domain and target the spaces $\{0\}$ if $r \geq 1$.

To sum up

$$H^p(\mathfrak{g}) = E_\infty^{0,p}(\mathfrak{g}) \quad \forall p = 0, \dots, \dim \mathfrak{g}.$$

Hence for each i , there is one intermediate cohomology group of degree i , namely $E_\infty^{0,i}$ which coincides with the usual Lie algebra cohomology.

Intermediate cohomology of the Heisenberg Lie algebra of dimension three.

Let $\{e_1, e_2, e_3\}$ be a basis of \mathfrak{h}_3 where the only non zero bracket is $[e_1, e_2] = e_3$. This is a 2-step nilpotent Lie algebra and, in the dual basis $\{e^1, e^2, e^3\}$ of \mathfrak{h}_3^* , the differential is $d(e^1) = d(e^2) = 0$ and $d(e^3) = -e^1 \wedge e^2$. Hence $V_0 = \{0\}$, $V_1 = \langle \{e^1, e^2\} \rangle$ and $V_2 = \mathfrak{h}_3^*$. The subcomplex $F^2 C^*$ is the zero complex, $F^0 C^*$ is the full Chevalley-Eilenberg complex $0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{h}_3^* \longrightarrow \Lambda^2 \mathfrak{h}_3^* \longrightarrow \Lambda^3 \mathfrak{h}_3^* \longrightarrow 0$ and there is also a complex $F^1 C^*$, such that

$$F^1 C^* : \quad 0 \longrightarrow \mathbb{R} \longrightarrow V_1 \longrightarrow \Lambda^2 V_1 \longrightarrow 0.$$

As before $E_0^{p,q} = \{0\}$ except for $p = 0, 1$, since $k = 2$ and Equation (6) implies

$$E_0^{0,0} = 0, \quad E_0^{0,q} = \frac{\Lambda^q \mathfrak{h}_3^*}{\Lambda^q V_1} \quad q = 1, 2, 3, \quad E_0^{1,-1} = \mathbb{R}, \quad E_0^{1,q} = \Lambda^{q+1} V_1 \quad q = 0, 1.$$

The differentials of the spectral sequence when $r = 0$ are vertical, implying that E_1 is, for $p = 1$, the cohomology of the complex $F^1 C^*$ and, for $p = 0$ the cohomology of the quotient complex $F^0 C^* / F^1 C^*$. Applying Formula (7) to obtain E_1 , one has

$$E_1^{0,0} = 0, \quad E_1^{0,1} = \frac{\mathfrak{h}_3^*}{V_1}, \quad E_1^{0,2} = \frac{\{x \in \Lambda^2 \mathfrak{h}_3^* : dx \in \Lambda^3 V_1 = \{0\}\}}{d(\{x \in \mathfrak{h}_3^* : dx \in \Lambda^2 \mathfrak{h}_3^*\}) + \{x \in \Lambda^2 V_1 : dx \in \Lambda^3 V_1\}} = \frac{\Lambda^2 \mathfrak{h}_3^*}{\Lambda^2 V_1}$$

where

$$\frac{\Lambda^2 \mathfrak{h}_3^*}{\Lambda^2 V_1} = \text{span}\{e^1 \wedge e^3 + \Lambda^2 V_1, e^2 \wedge e^3 + \Lambda^2 V_1\}.$$

Moreover,

$$E_1^{0,3} = \Lambda^3 \mathfrak{h}_3^*, \quad E_1^{1,-1} = \mathbb{R}, \quad E_1^{1,0} = V_1, \quad E_1^{1,1} = \Lambda^2 V_1.$$

Notice that E_1 coincides with E_0 . Again, from Equation (7), the term E_2 is

$$E_2^{0,0} = 0, \quad E_2^{0,1} = \frac{\{x \in \mathfrak{h}_3^* : dx = 0\}}{d(\mathfrak{h}_3^*) + \{x \in V_1 : dx = 0\}} = \frac{V_1}{V_1} \cong \{0\},$$

$$E_2^{0,2} = \frac{\{x \in \Lambda^2 \mathfrak{h}_3^* : dx = 0\}}{d(\mathfrak{h}_3^*) + \{x \in \Lambda^2 V_1 : dx = 0\}} = \frac{\Lambda^2 \mathfrak{h}_3^*}{\Lambda^2 V_1}, \quad E_2^{0,3} = \Lambda^3 \mathfrak{h}_3^*,$$

$$E_2^{1,-1} = \mathbb{R}, \quad E_2^{1,0} = V_1, \quad E_2^{1,1} = \frac{\{x \in \Lambda^2 V_1 : dx = 0\}}{d(\{x \in \mathfrak{h}_3^* : dx \in \Lambda^2 V_1\})} \cong \{0\}.$$

This is the limit term of the sequence since all differentials $d_2^{p,q} : E_2^{p,q} \longrightarrow E_2^{p+2,q-1}$ have as domain and target spaces the null space. Therefore

$$H^0(\mathfrak{h}_3) = \mathbb{R} \quad H^1(\mathfrak{h}_3) \cong E_2^{0,1} \oplus E_0^{1,0} \cong V_1 \quad H^2(\mathfrak{h}_3) \cong E_2^{0,2} \oplus E_2^{1,1} \cong \frac{\Lambda^2 \mathfrak{h}_3^*}{\Lambda^2 V_1}$$

$$H^3(\mathfrak{h}_3) = \Lambda^3 \mathfrak{h}_3^*.$$

Observe that in this example, fixed $i \in \{0, \dots, 3\}$, there are two intermediate cohomology groups of degree i , namely $E_\infty^{0,i}$ and $E_\infty^{1,i-1}$. Depending on i , one of those is zero and the other is the cohomology group $H^i(\mathfrak{g})$.

Remark 1. From Equation (9) one expects the intermediate cohomology of \mathfrak{g} to graduate the Lie algebra cohomology. Notice that in this does not happen in the previous examples. At the end of this section we present in Example 3.8 a Lie algebra whose cohomology groups are direct sum of non-trivial intermediate cohomology groups.

3.2. General Properties. Computing the terms of the spectral sequence associated to a nilpotent Lie algebra \mathfrak{g} is usually quite complicated. For this reason we introduce some properties that facilitate this work. Indeed, the theorem below explicits the intermediate cohomology group of degree 0, 1, $\dim \mathfrak{g} - 1$ and $\dim \mathfrak{g}$ which are mostly zero. Afterwards we establish some facts about the degeneracy of the spectral sequence. Finally, an analogous result to the well known Künneth formula is proved.

Theorem 3.2. *Let \mathfrak{g} be an m dimensional k -step nilpotent Lie algebra and let E_r its canonical spectral sequence with limit term E_∞ . Then*

- (1) $E_\infty^{k-1,1-k} = H^0(\mathfrak{g}) = \mathbb{R}$ and hence $E_\infty^{p,-p}(\mathfrak{g}) = 0$ for all $p = 0, \dots, k-2$.
- (2) $E_\infty^{k-1,2-k} = H^1(\mathfrak{g}) = \ker d$ and hence $E_\infty^{p,1-p}(\mathfrak{g}) = 0$ for all $p = 0, \dots, k-2$.
- (3) $E_\infty^{0,m-1} = H^{m-1}(\mathfrak{g})$ and hence $E_\infty^{p,m-1-p} = 0$ for all $p = 1, \dots, k-1$.
- (4) $E_\infty^{0,m} \cong H^m(\mathfrak{g}) \cong \mathbb{R}$ and hence $E_\infty^{p,m-p} = 0$ for all $p = 1, \dots, k-1$.

Proof. The first two assertions follow straight from the definition of the canonical spectral sequence of \mathfrak{g} . Moreover, it is well known that the top cohomology group $H^m(\mathfrak{g})$ of a nilpotent Lie algebra \mathfrak{g} is generated by a class of the form $e^1 \wedge \dots \wedge e^m$ if $\{e^1, \dots, e^m\}$ is a basis of \mathfrak{g}^* ([9]). This implies $E_\infty^{0,m} \cong H^m(\mathfrak{g}) \cong \mathbb{R}$ which combined with Equation (9) proves (4).

To prove (3), we make use of the following Lemma.

Lemma 3.3. *Let \mathfrak{g} be an m dimensional $(r+1)$ -step nilpotent Lie algebra, let n_0 be the dimension of $V_1 = \ker d$ and denote $\beta_1, \beta_2, \dots, \beta_{n_0}$ a basis of V_1 . Then every $\sigma \in \Lambda^{m-1} \mathfrak{g}^*$ is closed. Moreover σ is exact if and only if it is divisible by $\beta_1 \wedge \beta_2 \wedge \dots \wedge \beta_{n_0}$.*

Remark. This lemma joins a result due to Benson and Gordon in [1] and its reciprocal result made by Yamada in [17]. In both works, the result is proved under the hypothesis that \mathfrak{g} is a symplectic Lie algebra. Nevertheless we noticed that this is valid for any nilpotent Lie algebra. We include the proof here to make this fact explicit and for completeness of the exposition.

Proof of Lemma 3.3. Let \mathfrak{a}^i denote a vector space complementary to \mathfrak{g}^{i+1} in \mathfrak{g}^i where \mathfrak{g}^i is the i -th term of the central descending series of \mathfrak{g} . For $i = 0, 1, \dots, r$

$$\mathfrak{g}^i = \mathfrak{g}^{i+1} + \mathfrak{a}^i;$$

define $n_i = \dim \mathfrak{a}^i$. Denote

$$\Lambda^{i_0, i_1, \dots, i_r} = \Lambda^{i_0}(\mathfrak{a}^0)^* \wedge \Lambda^{i_1}(\mathfrak{a}^1)^* \wedge \dots \wedge \Lambda^{i_r}(\mathfrak{a}^r)^* \subseteq \Lambda^{i_0+i_1+\dots+i_r} \mathfrak{g}^*.$$

Then for $s = 0, \dots, m$

$$\Lambda^s \mathfrak{g}^* = \sum_{i_0+i_1+\dots+i_r=s} \Lambda^{i_0, i_1, \dots, i_r}.$$

Notice that it is possible to choose \mathfrak{a}^0 such that $(\mathfrak{a}^0)^* = V_1$. Let $\beta_1, \beta_2, \dots, \beta_{n_0}$ be a basis of $(\mathfrak{a}^0)^*$.

Assume $\eta \in \Lambda^{i_0, \dots, i_r}$ then each term of $d\eta$ belongs to a subspace $\Lambda^{t_0, t_1, \dots, t_r}$ and there exists an index $j \geq 1$ such that $t_j = i_j - 1$ and $t_0 + t_1 + \dots + t_{j-1} = i_0 + i_1 + \dots + i_{j-1} + 2$.

Suppose in particular that $\eta \in \Lambda^{m-1}\mathfrak{g}^*$, this implies $i_0 + i_1 + \dots + i_r = m - 1$. Then for any $j \geq 1$ we have $i_0 + i_1 + \dots + i_{j-1} \geq n_0 + n_1 + \dots + n_{j-1} - 1$. In fact if for some j holds $i_0 + i_1 + \dots + i_{j-1} < n_0 + n_1 + \dots + n_{j-1} - 1$, then $\sum_{l=1}^r i_l < \sum_{l=1}^r n_l - 1 = 2m - 1$ which leads to a contradiction. Fixed the term of $d\eta$ in $\Lambda^{t_0, t_1, \dots, t_r}$, let j be the index specified in the previous paragraph. It verifies $t_0 + t_1 + \dots + t_{j-1} = i_0 + i_1 + \dots + i_{j-1} + 2 > n_0 + n_1 + \dots + n_{j-1}$ implying $d\eta = 0$. Therefore any $m - 1$ form is closed.

Let $\sigma = d\alpha$ be an $m - 1$ exact form. Clearly each component η of α belongs to some $\Lambda^{i_0, \dots, i_r}$ with $i_0 + \dots + i_r = m - 2$. As before, the term of $d\eta$ belonging to $\Lambda^{t_0, t_1, \dots, t_r}$ satisfies $t_0 + t_1 + \dots + t_j = n_0 + n_1 + \dots + n_j$ for all $j \geq 1$. In particular, $t_0 = n_0$ which implies that σ is divisible by $\beta_0 \wedge \dots \wedge \beta_{n_0}$.

To prove that any exact $m - 1$ form is divisible by $\beta_0 \wedge \dots \wedge \beta_{n_0}$ a dimensional argument is used. Denote B^{m-1} the set of $m - 1$ exact forms, then $B^{m-1} \subseteq \sum_{n_0 + i_1 + \dots + i_r = m-1} \Lambda^{n_0, i_1, \dots, i_r}$. From Poincaré duality $\dim H^{m-1}(\mathfrak{g}) = \dim H^1(\mathfrak{g}) = n_0$. Moreover, the result above states $\dim H^{m-1}(\mathfrak{g}) = \dim \Lambda^{m-1}\mathfrak{g}^* - \dim B^{m-1}$. Combining these two formulas one leads to $\dim B^{m-1} = n_1 + \dots + n_r$ and therefore $B^{m-1} = \sum_{n_0 + i_1 + \dots + i_r = m-1} \Lambda^{n_0, i_1, \dots, i_r}$. ■

Proof of (3) in Theorem 3.2. From Equation (8) above and the previous Lemma,

$$E_{\infty}^{0, m-1} = \frac{\{x \in \Lambda^{m-1}\mathfrak{g}^* : dx = 0\}}{d(\{x \in \Lambda^{m-2}\mathfrak{g}^* : dx \in \Lambda^{m-1}\mathfrak{g}^*\}) + \{x \in \Lambda^{m-1}V_{k-1} : dx = 0\}} = \frac{\Lambda^{m-1}\mathfrak{g}^*}{B^{m-1} + \Lambda^{m-1}V_{k-1}}.$$

It is sufficient to show that $\Lambda^{m-1}V_{k-1} \subseteq B^{m-1}$ since it implies $E_{\infty}^{0, m-1} = \frac{\Lambda^{m-1}\mathfrak{g}^*}{B^{m-1}} = H^{m-1}(\mathfrak{g})$.

Notice that if $\dim V_{k-1} < m - 1$ then $\Lambda^{m-1}V_{k-1} = 0$ and it is clearly contained in B^{m-1} . When $\dim V_{k-1} = m - 1$, $\dim \mathfrak{a}^{k-1} = 1$ and the subspace $\Lambda^{m-1}V_{k-1}$ coincides with $\Lambda^{n_0, n_1, \dots, n_{k-2}, 0}$ and it is contained in B^{m-1} because of the Lemma above. Thus $E_{\infty}^{0, m-1} = H^{m-1}(\mathfrak{g})$ in any case.

By using Equation (9) one concludes that $E_{\infty}^{p, m-1-p} = 0$ for all $1 \leq p \leq k - 1$. ■

Below we analyze the behavior of the the number r_0 for which the spectral sequence degenerates, i.e. r_0 is the minimum r for which $E_r = E_{\infty}$. It is related to the nilpotency index and the dimension of the nilpotent Lie algebra \mathfrak{g} .

When the Lie algebra \mathfrak{g} is k -step nilpotent it is easy to see that for any $r \geq k$, Equations (7) and (8) coincide. Hence, the minimum r_0 is at most k . Namely,

Proposition 3.4. *Let \mathfrak{g} be a k -step nilpotent Lie algebra and E_r the canonical spectral sequence. For any $r \geq k$ it holds $E_r = E_{\infty}$.*

The purpose of the next example is to show that it is not possible to find a common r_0 valid for all nilpotent Lie algebras.

Example 3.5. For any $m \in \mathbb{N}$, there exists a nilpotent Lie algebra \mathfrak{g} of dimension $2m$ such that its canonical spectral sequence E_r does not degenerate in the $m - 1$ step, that is, $E_{\infty} \neq E_{m-1}$.

Let $n \in \mathbb{N}$, denote $\mathfrak{m}_0(n)$ the real Lie algebra with a basis $\{e_1, \dots, e_n\}$ and non zero brackets

$$[e_1, e_i] = -e_{i+1}, \quad 2 \leq i \leq n - 1.$$

This is a family of filiform Lie algebras and its cohomology is studied in the work of D. Millionschikov [11]. According to Lemma 5.2 there, the cohomology group $H^2(\mathfrak{m}_0(n))$ has

dimension $\left\lceil \frac{n+1}{2} \right\rceil$ for $n \geq 3$ and it has a basis given by the cohomology classes of the cocycles

$$e^1 \wedge e^n, \left\{ \frac{1}{2} \sum_{i=2}^{2k-1} (-1)^i e^i \wedge e^{2k+1-i}, \quad k = 2, \dots, \left\lceil \frac{n+1}{2} \right\rceil \right\}.$$

Notice that for $n = 6$, $\dim H^2(\mathfrak{m}_0(6)) = 3$ and a basis of that space is given by the cohomology classes of

$$e^1 \wedge e^6, e^2 \wedge e^3, e^2 \wedge e^5 - e^3 \wedge e^4.$$

Moreover, $E_{\infty}^{0,2}(\mathfrak{m}_0(6))$ has dimension one and it is generated by $e^1 \wedge e^6$.

This fact can be easily generalized to any even dimension, actually one proves that for $m \geq 2$ it holds

$$(10) \quad \dim E_{\infty}^{0,2}(\mathfrak{m}_0(2m)) = 1.$$

More specifically, it is generated by the class of $e^1 \wedge e^{2m}$.

We show below that $E_{m-1}^{0,2}(\mathfrak{m}_0(2m))$ has dimension greater than 2. This fact together with (10) implies that $E_{m-1} \neq E_{\infty}$, as we wanted to show.

Denote $\mathfrak{g} := \mathfrak{m}_0(2m)$, its differential $d : \mathfrak{g}^* \rightarrow \Lambda^2 \mathfrak{g}^*$ is determined by the non zero brackets defining $\mathfrak{m}_0(2m)$. Therefore

$$(11) \quad de^i = \begin{cases} 0 & \text{if } i = 1, 2, \\ e^1 \wedge e^{i-1} & \text{if } i = 3, \dots, 2m \end{cases}.$$

Notice that the filtration of \mathfrak{g}^* is $V_i = \text{span}\{e_1, \dots, e_{i+1}\}$ for $i = 1, \dots, 2m-1$.

Define

$$\sigma = \sum_{j=0}^{m-2} (-1)^j e^{j+2} \wedge e^{2m-j} = e^2 \wedge e^{2m} - e^3 \wedge e^{2m-1} + \dots + (-1)^{m-2} e^m \wedge e^{m+2}.$$

This is an element in $\Lambda^2 \mathfrak{g}^*$ not belonging to $\Lambda^2 V_{2m-2}$. By Equation (11) its differential is

$$d\sigma = (-1)^{m-2} e^1 \wedge e^m \wedge e^{m+1}.$$

Thus $d\sigma \in \Lambda^2 V_m$ since $V_m = \text{span}\{e^1, \dots, e^{m+1}\}$.

In the $(m-1)$ -th term of the spectral sequence we have

$$E_{m-1}^{0,2} = \frac{\{x \in \Lambda^2 \mathfrak{g}^* : dx \in \Lambda^3 V_m\}}{\{x \in \Lambda^2 V_{2m-2} : dx \in \Lambda^3 V_m\}}.$$

In fact since $d\mathfrak{g}^* \subseteq \Lambda^2 V_{2m-1}$, Equation (7) implies the previous formula.

Hence σ defines a non zero element in $E_{m-1}^{0,2}$. Also $e^1 \wedge e^{2m}$ defines a non zero element in $E_{m-1}^{0,2}$ which is linearly independent to that one defined by σ . Hence $\dim E_{m-1}^{0,2}(\mathfrak{m}_0(2m)) \geq 2$.

Remark. It would be interesting to find numbers $r_0(m)$ for which the spectral sequence of a Lie algebra \mathfrak{g} of dimension m degenerates at $r \leq r_0(m)$. As a consequence of the last example this number $r_0(m)$ must be at least $m/2$. We have examples to believe that $r_0(m)$ would be $m/2$.

In Lie algebra cohomology the well known Künneth formula relates the Lie algebra cohomology of a Lie algebra which is a direct sum of ideals, with the cohomology of its summands.

When the nilpotent Lie algebra \mathfrak{g} can be decomposed as a direct sum of ideals \mathbb{R} and a nilpotent Lie algebra of dimension one less than \mathfrak{g} , a similar formula can be proved for intermediate cohomology.

Theorem 3.6. *Let \mathfrak{g} be a k -step nilpotent Lie algebra which can be decomposed as a direct sum of ideals $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{h}$. Then \mathfrak{h} is k -step nilpotent and $\forall r \geq 0$ and $r = \infty$*

- (1) $E_r^{p,-p}(\mathfrak{g}) = 0$ for all $p = 0, \dots, k-2$ and $E_r^{k-1,1-k}(\mathfrak{g}) \cong \mathbb{R}$.
- (2) $E_r^{k-1,2-k}(\mathfrak{g}) \cong E_r^{k-1,2-k}(\mathfrak{h}) \oplus \mathbb{R}$,
- (3) $E_r^{p,1-p}(\mathfrak{g}) \cong E_r^{p,1-p}(\mathfrak{h})$ if $p \leq k-2$,
- (4) $E_r^{p,q}(\mathfrak{g}) \cong E_r^{p,q}(\mathfrak{h}) \oplus E_r^{p,q-1}(\mathfrak{h})$ if $p+q \geq 2$.

Proof. We will only show some equalities that lead to the complete proof.

The first assertion comes from (1) of Theorem 3.2. Suppose $\mathfrak{g} = \mathbb{R}x \oplus \mathfrak{h}$. Denote by x^* the element in \mathfrak{g}^* such that $x^*(x) = 1$, $x^*(\mathfrak{h}) = 0$ and identify \mathfrak{h}^* with a subset of \mathfrak{g}^* . Clearly $dx^* = 0$ and \mathfrak{h}^* is invariant by d . Also the restriction of d to \mathfrak{h}^* is the differential of the Lie algebra \mathfrak{h} .

The subsets $V_0 \subseteq V_1 \subseteq \dots \subseteq V_k = \mathfrak{g}^*$ and $\tilde{V}_0 \subseteq \tilde{V}_1 \subseteq \dots \subseteq \tilde{V}_k = \mathfrak{h}^*$ which filter \mathfrak{g}^* and \mathfrak{h}^* respectively satisfy:

$$\tilde{V}_0 = V_0 = 0, \quad V_i = \tilde{V}_i \oplus \mathbb{R}x^*, \quad i = 1, \dots, k.$$

Equation (6) gives the initial term of $E_r(\mathfrak{g})$. For total degree 1, $E_0^{p,1-p}(\mathfrak{g}) = \frac{V_{k-p}}{V_{k-p-1}}$ and hence

$$E_0^{p,1-p}(\mathfrak{g}) = \begin{cases} (\tilde{V}_{k-p} \oplus \mathbb{R}x^*)/(\tilde{V}_{k-p-1} \oplus \mathbb{R}x^*) \cong \tilde{V}_{k-p}/\tilde{V}_{k-p-1} = E_0^{p,1-p}(\mathfrak{h}) & \text{if } p \neq k-1 \\ V_1 = \tilde{V} \oplus \mathbb{R}x^* & \text{if } p = k-1 \end{cases}.$$

When the total degree is greater or equal than 2 then $\Lambda^{p+q}V_{k-p} = \Lambda^{p+q}\tilde{V}_{k-p} \oplus (\mathbb{R}x^* \wedge \Lambda^{p+q-1}\tilde{V}_{k-p})$. Since $E_0^{p,q}(\mathfrak{g}) = \Lambda^{p+q}V_{k-p}/\Lambda^{p+q}V_{k-p-1}$,

$$\begin{aligned} E_0^{p,q} &= \frac{\Lambda^{p,q}\tilde{V}_{k-p} \oplus \mathbb{R}x^* \wedge \Lambda^{p+q-1}\tilde{V}_{k-p}}{\Lambda^{p,q}\tilde{V}_{k-p-1} \oplus \mathbb{R}x^* \wedge \Lambda^{p+q-1}\tilde{V}_{k-p-1}} \cong \frac{\Lambda^{p,q}\tilde{V}_{k-p}}{\Lambda^{p,q}\tilde{V}_{k-p-1}} \oplus \frac{\Lambda^{p+q-1}\tilde{V}_{k-p}}{\Lambda^{p+q-1}\tilde{V}_{k-p-1}} \\ &\cong E_0^{p,q}(\mathfrak{h}) \oplus E_0^{p,q-1}(\mathfrak{h}). \end{aligned}$$

For $r \geq 1$, Equation (7) implies

$$(12) \quad E_r^{k-1,q}(\mathfrak{g}) = \frac{\{y \in \Lambda^{k+q-1}V_1 : dy = 0\}}{d(\{y \in \Lambda^{k+q-2}V_r : dy \in \Lambda^{k+q-1}V_1\})}.$$

If $q = 2 - k$ then $E_r^{k-1,2-k}(\mathfrak{g}) = \{x \in V_1 : dy = 0\} = V_1 = \tilde{V}_1 \oplus \mathbb{R}x^* \cong E_r^{k-1,2-k}(\mathfrak{h}) \oplus \mathbb{R}$. When $q \geq 3 - k$ ($\Leftrightarrow k + q - 1 \geq 2$)

$$\Lambda^{k+q-1}V_1 = \Lambda^{k+q-1}\tilde{V}_1 \oplus \mathbb{R}x^* \wedge \Lambda^{k+q-2}\tilde{V}_1,$$

therefore $\omega \in \Lambda^{k+q-1}V_1 \Leftrightarrow \omega = \omega_1 + x^* \wedge \omega_2$ with $\omega_1 \in \Lambda^{k+q-1}\tilde{V}_1$, $\omega_2 \in \Lambda^{k+q-2}\tilde{V}_1$. Moreover $d\omega = 0 \Leftrightarrow d\omega_1 + x^* \wedge d\omega_2 = 0 \Leftrightarrow d\omega_1 = d\omega_2 = 0$. Then the numerator in (12) is written as

$$\{y \in \Lambda^{k+q-1}V_1 : dy = 0\} = \{x \in \Lambda^{k+q-1}\tilde{V}_1 : dy = 0\} \oplus \mathbb{R}x^* \wedge \{x \in \Lambda^{k+q-2}\tilde{V}_1 : dy = 0\}.$$

To describe the denominator it is necessary to consider two cases: $k + q - 2 = 1$ ($\Leftrightarrow q = 3 - k$) and $k + q - 2 \geq 2$. In the first case

$$d(\{y \in \Lambda^{k+q-2}V_r : dy \in \Lambda^{k+q-1}V_1\}) = d(\{y \in V_r : dy \in \Lambda^2V_1\}) = d(V_2) = d(\tilde{V}_2),$$

having

$$\begin{aligned} E_r^{k-1,3-k}(\mathfrak{g}) &= \frac{\{x \in \Lambda^2\tilde{V}_1 : dy = 0\} \oplus \mathbb{R}x^* \wedge \{x \in \tilde{V}_1 : dy = 0\}}{d(\tilde{V}_2)} \\ &\cong E_r^{k-1,3-k}(\mathfrak{h}) \oplus \underbrace{\mathbb{R}x^* \wedge \tilde{V}_1}_{\cong \tilde{V}_1} \cong E_r^{k-1,3-k}(\mathfrak{h}) \oplus E_r^{k-1,2-k}(\mathfrak{h}). \end{aligned}$$

In the case $k + q - 2 \geq 2$ ($\Leftrightarrow q \geq 4 - k$) $\Lambda^{k+q-2}V_r = \Lambda^{k+q-2}\tilde{V}_r \oplus (\mathbb{R}x^* \wedge \Lambda^{k+q-3}\tilde{V}_r)$. Then, every $\omega \in \Lambda^{k+q-2}V_r$ can be written as $\omega = \omega_1 + x^* \wedge \omega_2$ where $\omega_1 \in \Lambda^{k+q-2}\tilde{V}_r$ and $\omega_2 \in \Lambda^{k+q-3}\tilde{V}_r$. Moreover $\Lambda^{k+q-1}V_1 = \Lambda^{k+q-1}\tilde{V}_1 \oplus \mathbb{R}x^* \wedge \Lambda^{k+q-2}\tilde{V}_1$.

Hence for $\omega \in \Lambda^{k+q-2}V_r$

$$d\omega = d\omega_1 + x^* \wedge d\omega_2 \in \Lambda^{k+q-1}V_1 \text{ if and only if } d\omega_1 \in \Lambda^{k+q-1}\tilde{V}_1 \text{ and } d\omega_2 \in \Lambda^{k+q-2}\tilde{V}_1.$$

Therefore

$$\begin{aligned} d(\{y \in \Lambda^{k+q-2}V_r : dy \in \Lambda^{k+q-1}V_1\}) &= \\ &= d(\{y \in \Lambda^{k+q-2}\tilde{V}_r : dy \in \Lambda^{k+q-1}\tilde{V}_1\}) \oplus \mathbb{R}x^* \wedge d(\{y \in \Lambda^{k+q-3}\tilde{V}_r : dy \in \Lambda^{k+q-2}\tilde{V}_1\}). \end{aligned}$$

Combining the formulas in the numerator and denominator, Equation (12) becomes

$$E_r^{k-1,q}(\mathfrak{g}) \cong E_r^{k-1,q}(\mathfrak{h}) \oplus E_r^{k-1,q-1}(\mathfrak{h}).$$

For those $p \neq k - 1$ the proof is very similar. ■

Applying an inductive procedure, one proves a similar formula for Lie algebras which has an abelian direct factor of dimension s , $s \geq 1$.

Corollary 3.7. *Let \mathfrak{g} be a nilpotent Lie algebra that decomposes as a direct sum of ideals $\mathbb{R}^s \oplus \mathfrak{h}$ for some $s \geq 1$. Then each term of the spectral sequence $E_r(\mathfrak{g})$ can be written as a sum of certain terms of the spectral sequence $E_r(\mathfrak{h})$. In particular $E_r(\mathfrak{g})$ degenerates at r_0 if and only if $E_r(\mathfrak{h})$ does.*

Example 3.8. [Application of Theorem 3.6.] Let \mathfrak{g} be the four dimensional Lie algebra with basis $\{e_1, e_2, e_3, e_4\}$ and non zero bracket $[e_2, e_3] = e_4$. This algebra is two step nilpotent and it is isomorphic to $\mathbb{R} \oplus \mathfrak{h}_3$. Recall the intermediate cohomology of the Heisenberg Lie algebra computed in Section 3. As stated in (1) of Proposition 2.2, $E_\infty^{p,q} = 0$ if $p < 0$ or $p \geq 2$. Using the formulas in the previous proposition, for $r = \infty$, we have

- (1) $E_\infty^{0,0}(\mathfrak{g}) = 0$, $E_\infty^{1,-1}(\mathfrak{g}) = \mathbb{R}$,
- (2) $E_\infty^{1,0}(\mathfrak{g}) \cong E_\infty^{1,0}(\mathfrak{h}) \oplus \mathbb{R} = \text{span}\{e^2, e^3\} \oplus \mathbb{R}$,
- (3) $E_\infty^{0,1}(\mathfrak{g}) \cong E_\infty^{0,1}(\mathfrak{h}) = 0$,
- (4) - $E_\infty^{0,2}(\mathfrak{g}) \cong E_\infty^{0,2}(\mathfrak{h}) \oplus E_\infty^{0,1}(\mathfrak{h}) \cong \text{span}\{e^2 \wedge e^4, e^3 \wedge e^4\}$,
 - $E_\infty^{1,1}(\mathfrak{g}) \cong E_\infty^{1,1}(\mathfrak{h}) \oplus E_\infty^{1,0}(\mathfrak{h}) \cong \text{span}\{e^2, e^3\}$,
 - $E_\infty^{0,3}(\mathfrak{g}) \cong E_\infty^{0,3}(\mathfrak{h}) \oplus E_\infty^{0,2}(\mathfrak{h}) \cong \text{span}\{e^2 \wedge e^3 \wedge e^4\} \oplus \text{span}\{e^2 \wedge e^4, e^3 \wedge e^4\}$,
 - $E_\infty^{1,2}(\mathfrak{g}) \cong E_\infty^{1,2}(\mathfrak{h}) \oplus E_\infty^{1,1}(\mathfrak{h}) = 0$ since $E_\infty^{1,1}(\mathfrak{h}) = E_\infty^{1,2}(\mathfrak{h}) = 0$,
 - $E_\infty^{0,4}(\mathfrak{g}) \cong E_\infty^{0,4}(\mathfrak{h}) \oplus E_\infty^{0,3}(\mathfrak{h}) \cong \text{span}\{e^2 \wedge e^3 \wedge e^4\}$.

Hence, the cohomology of the Lie algebra decomposes as follows

$$\begin{aligned}
 H^0(\mathfrak{g}) &\cong E_{\infty}^{0,0}(\mathfrak{g}) = \mathbb{R}, \\
 H^1(\mathfrak{g}) &\cong E_{\infty}^{1,0}(\mathfrak{g}) \oplus E_{\infty}^{0,1}(\mathfrak{g}) \cong \text{span}\{e^2, e^3\} \oplus \mathbb{R}, \\
 H^2(\mathfrak{g}) &\cong E_{\infty}^{0,2}(\mathfrak{g}) \oplus E_{\infty}^{1,1}(\mathfrak{g}) \cong \text{span}\{e^2 \wedge e^4, e^3 \wedge e^4\} \oplus \text{span}\{e^2, e^3\} \\
 H^3(\mathfrak{g}) &\cong E_{\infty}^{0,3}(\mathfrak{g}) \oplus E_{\infty}^{1,2}(\mathfrak{g}) \cong \text{span}\{e^2 \wedge e^3 \wedge e^4\} \oplus \text{span}\{e^2 \wedge e^4, e^3 \wedge e^4\}, \\
 H^4(\mathfrak{g}) &\cong E_{\infty}^{0,4}(\mathfrak{g}) \cong \text{span}\{e^2 \wedge e^3 \wedge e^4\}.
 \end{aligned}$$

Observe that in degree 3, the intermediate cohomology groups are both non zero (compare with Remark 1)

4. INTERMEDIATE COHOMOLOGY DIAGRAMS

In this section we present a useful manner of displaying the intermediate cohomology of a k -step nilpotent Lie algebra \mathfrak{g} of dimension m instead of listing it as done in the examples above.

After computing the terms $E_0, E_1, \dots, E_r = E_{\infty}$ of the spectral sequence it is convenient to display this information inside r tables each of them having k rows and $m+1$ columns, that is, one for each term E_r . Fixed $j \in \{0, \dots, r\}$, the table corresponding to E_j shows in the first column the terms of total degree 0, in the second column the ones of total degree 1, and so on. More precisely the construction of the table E_j is as the one below.

$\dim E_j^{k-1,1-k}$	$\dim E_j^{k-1,2-k}$	$\dim E_j^{k-1,3-k}$	\dots	$\dim E_j^{k-1,m+1-k}$
\vdots	\vdots	\vdots	\dots	\vdots
$\dim E_j^{1,-1}$	$\dim E_j^{1,0}$	$\dim E_j^{1,1}$	\dots	$\dim E_j^{1,m-1}$
$\dim E_j^{0,0}$	$\dim E_j^{0,1}$	$\dim E_j^{0,2}$	\dots	$\dim E_j^{0,m}$

Some properties of the intermediate cohomology can be seen in these diagrams. From the results in Theorem 3.2, the first two columns and the last two columns consist of zeros except for the top or bottom numbers. Precisely, the first column (last column) has a number 1 at the top (bottom) of the table and the second column (penultimate column) has $\dim \ker d$ at the top (bottom) of the table.

From Equation (9), the sum of the elements of the i -th column in the table corresponding to the limit term gives as result the i -th Betti number of the Lie algebra \mathfrak{g} , i.e.

$$\beta_i = \dim H^i(\mathfrak{g}) = \sum_{p+q=i} \dim E_{\infty}^{p,q}.$$

For nilpotent Lie algebras the Poincaré duality holds (see [9]) hence, for the E_{∞} table, the sum of the elements of the i -th column coincides with the sum of the $(m-i)$ -th column.

The tables corresponding to the examples presented in the previous sections are:

Abelian Lie algebras Since $k = 1$ the tables have only one row and $m+1$ columns, being m the dimension of \mathfrak{g} . The table of E_0 has the dimensions of the spaces $\Lambda^{p+q}\mathfrak{g}^*$ and, as seen before, E_1 coincides with E_0 and it is the the cohomology of \mathfrak{g}^* .

$$E_0 = E_\infty$$

1	m	$\begin{pmatrix} m \\ 2 \end{pmatrix}$	\cdots	$\begin{pmatrix} m \\ m-1 \end{pmatrix}$	$\begin{pmatrix} m \\ m \end{pmatrix}$
---	-----	--	----------	--	--

Heisenberg Lie algebra of dimension 3 These tables have two rows since \mathfrak{h}_3 is two step nilpotent and 4 columns since this is the dimension of \mathfrak{h}_3 plus one.

$$E_0 \quad E_1 \quad E_2 = E_\infty$$

1	2	1	0	1	2	1	0	1	2	0	0
0	1	2	1	0	1	2	1	0	0	2	1

Lie algebra $\mathbb{R} \oplus \mathfrak{h}_3$ of dimension 4 This is the Lie algebra in Example 3.8. The tables have two rows and five columns.

$$E_0 \quad E_1 \quad E_2 = E_\infty$$

1	3	3	1	0	1	3	3	1	0	1	3	2	0	0
0	1	3	3	1	0	1	3	3	1	0	0	2	3	1

Observe that even though $E_0 = E_1$ this is not the limit of the spectral sequence.

Theorem 3.6 gives a way to construct the tables of intermediate cohomology of a Lie algebra of the form $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{h}$ once we know the tables of \mathfrak{h} . If $e_r^{p,q}$ denotes the dimension of $E_r^{p,q}(\mathfrak{h})$ then the table corresponding to $E_r(\mathfrak{g})$ is

1	$e_r^{k-1,2-k} + 1$	$e_r^{k-1,3-k} + e_r^{k-1,2-k}$	$e_r^{k-1,4-k} + e_r^{k-1,3-k}$	\cdots	0	0
\vdots	\vdots	\vdots	\vdots	\cdots	\vdots	\vdots
0	0	$e_r^{1,1} + e_r^{1,0}$	$e_r^{1,2} + e_r^{1,1}$	\cdots	0	0
0	0	$e_r^{0,2} + e_r^{0,1}$	$e_r^{0,3} + e_r^{0,2}$	\cdots	$e_r^{0,m-1} + e_r^{0,m-2}$	1

5. INTERMEDIATE COHOMOLOGY IN LOW DIMENSIONS

Nilpotent Lie algebras over \mathbb{R} are classified up to dimension seven. Though six is the highest dimension in which there do not exist continuous families [10, 8]. For this reason we compute explicitly the intermediate cohomology of nilpotent Lie algebras up to dimension six. The information is displayed in tables as explained in the previous section.

In dimension one and two the only nilpotent Lie algebras are the abelian ones. In dimension three there is only one non abelian nilpotent Lie algebra that is the Heisenberg Lie algebra. The intermediate cohomology in those cases was exposed previously in this work.

The computation of the spectral sequence and the intermediate cohomology groups of nilpotent Lie algebras was made by hand and checked with a computational program we developed. Below we show the results we obtained for nilpotent Lie algebras of dimension at most six. We want to remark that in dimension six they are ordered as in [16].

The notation we use to describe the Lie algebras is the usual. A Lie algebra denoted as $\mathfrak{g} = (0, 0, 12, 13, 23, 14 + 25)$ is that one which has a basis $\{e^1, e^2, e^3, e^4, e^5, e^6\}$ of 1-forms where the differential is $de^1 = de^2 = 0$ and

$$de^3 = e^1 \wedge e^2 \quad de^4 = e^1 \wedge e^3 \quad de^5 = e^2 \wedge e^3 \quad de^6 = e^1 \wedge e^4 + e^2 \wedge e^5.$$

According to the Maurer-Cartan formula, the non zero Lie brackets of this Lie algebra are

$$[e_1, e_2] = -e_3, \quad [e_1, e_3] = -e_4, \quad [e_2, e_3] = -e_5 \quad [e_1, e_4] = -e_6 = [e_2, e_5],$$

if $\{e_i : i = 1, \dots, 6\}$ is the dual basis of $\{e^i : i = 1, \dots, 6\}$.

We also specify when the Lie algebra listed \mathfrak{g} admits a decomposition in direct sum of ideals of the form $\mathfrak{g} = \mathbb{R}^s \oplus \mathfrak{h}$ for the reader to compare the tables of \mathfrak{g} and \mathfrak{h} according to Theorem 3.6.

Dimension four.

(1) $\mathfrak{g} = (0, 0, 12, 13)$

E_0					E_1					$E_2 = E_\infty$				
1	2	1	0	0	1	2	1	0	0	1	2	0	0	0
0	1	2	1	0	0	1	2	1	0	0	0	1	0	0
0	1	3	3	1	0	1	2	2	1	0	0	1	2	1

(2) $\mathfrak{g} = (0, 0, 0, 12)$.

E_0					E_1					$E_2 = E_\infty$				
1	3	3	1	0	1	3	3	1	0	1	3	2	0	0
0	1	3	3	1	0	1	3	3	1	0	0	2	3	1

Dimension five.

(1) $\mathfrak{g} = (0, 0, 12, 13, 14 + 23)$:

E_0						E_1						E_2					
1	2	1	0	0	0	1	2	1	0	0	0	1	2	0	0	0	0
0	1	2	1	0	0	0	1	2	1	0	0	0	0	1	0	0	0
0	1	3	3	1	0	0	1	2	2	1	0	0	0	0	2	0	0
0	1	4	6	4	1	0	1	2	2	2	1	0	0	2	1	2	1

(2) $\mathfrak{g} = (0, 0, 12, 13, 14)$:

E_0						E_1						E_2					
1	2	1	0	0	0	1	2	1	0	0	0	1	2	0	0	0	0
0	1	2	1	0	0	0	1	2	1	0	0	0	0	1	0	0	0
0	1	3	3	1	0	0	1	2	2	1	0	0	0	0	2	0	0
0	1	4	6	4	1	0	1	2	2	2	1	0	0	2	1	2	1

(3) $\mathfrak{g} = (0, 0, 12, 13, 23)$:

E_0						E_1						E_2					
1	2	1	0	0	0	1	2	1	0	0	0	1	2	0	0	0	0
0	1	2	1	0	0	0	1	2	1	0	0	0	0	0	0	0	0
0	2	7	9	5	1	0	1	4	3	2	1	0	0	3	3	2	1

(4) $\mathfrak{g} = (0, 0, 0, 12, 14 + 23)$:

E_0						E_1						E_2					
1	3	3	1	0	0	1	3	3	1	0	0	1	3	2	0	0	0
0	1	3	3	1	0	0	1	3	3	1	0	0	0	1	1	0	0
0	1	4	6	4	1	0	1	3	4	3	1	0	0	1	3	3	1

(5) $\mathfrak{g} = (0, 0, 0, 12, 24)$: $\mathfrak{g} = \mathbb{R}e_3 \oplus \mathfrak{h}$ where \mathfrak{h} is the 3-step nilpotent Lie algebra of dimension 4.

E_0						E_1						E_2					
1	3	3	1	0	0	1	3	3	1	0	0	1	3	2	0	0	0
0	1	3	3	1	0	0	1	3	3	1	0	0	0	1	1	0	0
0	1	4	6	4	1	0	1	3	4	3	1	0	0	1	3	3	1

(6) $\mathfrak{g} = (0, 0, 0, 12, 13)$:

E_0	E_1	E_2																																				
<table><tr><td>1</td><td>3</td><td>3</td><td>1</td><td>0</td><td>0</td></tr><tr><td>0</td><td>2</td><td>7</td><td>9</td><td>5</td><td>1</td></tr></table>	1	3	3	1	0	0	0	2	7	9	5	1	<table><tr><td>1</td><td>3</td><td>3</td><td>1</td><td>0</td><td>0</td></tr><tr><td>0</td><td>2</td><td>6</td><td>6</td><td>3</td><td>1</td></tr></table>	1	3	3	1	0	0	0	2	6	6	3	1	<table><tr><td>1</td><td>3</td><td>1</td><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>0</td><td>5</td><td>6</td><td>3</td><td>1</td></tr></table>	1	3	1	0	0	0	0	0	5	6	3	1
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(7) $\mathfrak{g} = (0, 0, 0, 0, 12)$: $\mathfrak{g} = \mathbb{R}e_3 \oplus \mathbb{R}e_4 \oplus \mathfrak{h}$ where \mathfrak{h} is the 2-step nilpotent Lie algebra of dimension 3.

E_0	E_1	E_2																																				
<table><tr><td>1</td><td>4</td><td>6</td><td>4</td><td>1</td><td>0</td></tr><tr><td>0</td><td>1</td><td>4</td><td>6</td><td>4</td><td>1</td></tr></table>	1	4	6	4	1	0	0	1	4	6	4	1	<table><tr><td>1</td><td>4</td><td>6</td><td>4</td><td>1</td><td>0</td></tr><tr><td>0</td><td>1</td><td>4</td><td>6</td><td>4</td><td>1</td></tr></table>	1	4	6	4	1	0	0	1	4	6	4	1	<table><tr><td>1</td><td>4</td><td>5</td><td>2</td><td>0</td><td>0</td></tr><tr><td>0</td><td>0</td><td>2</td><td>5</td><td>4</td><td>1</td></tr></table>	1	4	5	2	0	0	0	0	2	5	4	1
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(8) $\mathfrak{g} = (0, 0, 0, 0, 12 + 34)$:

E_0	E_1	E_2																																				
<table><tr><td>1</td><td>4</td><td>6</td><td>4</td><td>1</td><td>0</td></tr><tr><td>0</td><td>1</td><td>4</td><td>6</td><td>4</td><td>1</td></tr></table>	1	4	6	4	1	0	0	1	4	6	4	1	<table><tr><td>1</td><td>4</td><td>6</td><td>4</td><td>1</td><td>0</td></tr><tr><td>0</td><td>1</td><td>4</td><td>6</td><td>4</td><td>1</td></tr></table>	1	4	6	4	1	0	0	1	4	6	4	1	<table><tr><td>1</td><td>4</td><td>5</td><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>0</td><td>0</td><td>5</td><td>4</td><td>1</td></tr></table>	1	4	5	0	0	0	0	0	0	5	4	1
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1	4	5	0	0	0																																	
0	0	0	5	4	1																																	

Dimension six.

(1) $\mathfrak{g} = (0, 0, 12, 13, 14 + 23, 34 + 52)$:

E_0							E_1							E_2							$E_3 = E_\infty$						
1	2	1	0	0	0	0	1	2	1	0	0	0	0	1	2	0	0	0	0	0	1	2	0	0	0	0	0
0	1	2	1	0	0	0	0	1	2	1	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0
0	1	3	3	1	0	0	0	1	2	2	1	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0
0	1	4	6	4	1	0	0	0	1	2	2	2	1	0	0	0	1	1	0	0	0	0	0	1	1	0	0
0	1	5	10	10	5	1	0	0	1	2	3	3	2	1	0	0	2	1	2	2	1	0	0	0	1	2	2

(2) $\mathfrak{g} = (0, 0, 12, 13, 14, 34 + 52)$:

E_0							E_1							E_2							$E_3 = E_\infty$							
1	2	1	0	0	0	0	1	2	1	0	0	0	0	1	2	0	0	0	0	0	0	1	2	0	0	0	0	0
0	1	2	1	0	0	0	0	1	2	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0
0	1	3	3	1	0	0	0	0	1	2	2	1	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0
0	1	4	6	4	1	0	0	0	1	2	2	2	1	0	0	0	1	1	0	0	0	0	0	0	1	1	0	0
0	1	5	10	10	5	1	0	0	1	2	3	3	5	1	0	0	0	2	1	2	2	1	0	0	0	1	2	2

(3) $\mathfrak{g} = (0, 0, 12, 13, 14, 15)$:

E_0							E_1							E_2							$E_3 = E_\infty$								
1	2	1	0	0	0	0	1	2	1	0	0	0	0	1	2	0	0	0	0	0	0	1	2	0	0	0	0	0	0
0	1	2	1	0	0	0	0	1	2	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0
0	1	3	3	1	0	0	0	1	2	2	1	0	0	0	0	0	2	0	0	0	0	0	0	0	0	1	0	0	0
0	1	4	6	4	1	0	0	0	1	2	2	2	1	0	0	0	1	1	1	0	0	0	0	0	0	1	1	0	0
0	1	5	10	10	5	1	0	0	1	2	3	3	2	1	0	0	2	2	2	2	2	1	0	0	1	2	2	2	1

(4) $\mathfrak{g} = (0, 0, 12, 13, 14 + 23, 15 + 24)$:

E_0							E_1							E_2							$E_3 = E_\infty$							
1	2	1	0	0	0	0	1	2	1	0	0	0	0	1	2	0	0	0	0	0	0	1	2	0	0	0	0	0
0	1	2	1	0	0	0	0	1	2	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0
0	1	3	3	1	0	0	0	1	2	2	1	0	0	0	0	0	2	0	0	0	0	0	0	0	1	0	0	0
0	1	4	6	4	1	0	0	0	1	2	2	2	1	0	0	0	1	1	1	0	0	0	0	0	1	1	1	0
0	1	5	10	10	5	1	0	0	1	2	3	2	2	1	0	0	2	2	2	2	2	1	0	0	1	2	2	2

(5) $\mathfrak{g} = (0, 0, 12, 13, 14, 15 + 23)$:

E_0	E_1	E_2	$E_3 = E_\infty$																																																																																																																																												
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(6) $\mathfrak{g} = (0, 0, 12, 13, 23, 14)$:

E_0							E_1							$E_2 = E_\infty$								
1	2	1	0	0	0	0	1	2	1	0	0	0	0	0	1	2	0	0	0	0	0	0
0	1	2	1	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	0	0	0	0
0	2	7	9	5	1	0	0	2	4	3	2	1	0	0	0	2	3	2	0	0	0	0
0	1	5	10	10	5	1	0	0	1	2	3	3	2	1	0	0	2	3	2	2	1	0

(7) $\mathfrak{g} = (0, 0, 12, 13, 23, 14 - 25)$:

E_0							E_1							$E_2 = E_\infty$								
1	2	1	0	0	0	0	1	2	1	0	0	0	0	0	1	2	0	0	0	0	0	0
0	1	2	1	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	0	0	0	0
0	2	7	9	5	1	0	0	2	4	3	2	1	0	0	0	2	3	2	0	0	0	0
0	1	5	10	10	5	1	0	0	1	2	3	3	2	1	0	0	2	3	2	2	1	0

(8) $\mathfrak{g} = (0, 0, 12, 13, 23, 14 + 25)$:

E_0							E_1							$E_2 = E_\infty$								
1	2	1	0	0	0	0	1	2	1	0	0	0	0	0	1	2	0	0	0	0	0	0
0	1	2	1	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0	0	0	0	0
0	2	7	9	5	1	0	0	2	4	3	2	1	0	0	0	2	3	2	0	0	0	0
0	1	5	10	10	5	1	0	0	1	2	3	3	2	1	0	0	2	3	2	2	1	0

(9) $\mathfrak{g} = (0, 0, 0, 12, 14 - 23, 15 + 34)$:

E_0	E_1	E_2	$E_3 = E_\infty$																																																																																																																
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(10) $\mathfrak{g} = (0, 0, 0, 12, 14, 15 + 23)$:

E_0	E_1	$E_2 = E_\infty$																																																																																				
<table><tr><td>1</td><td>3</td><td>3</td><td>1</td><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>3</td><td>3</td><td>1</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>4</td><td>6</td><td>4</td><td>1</td><td>0</td></tr><tr><td>0</td><td>1</td><td>5</td><td>10</td><td>10</td><td>5</td><td>1</td></tr></table>	1	3	3	1	0	0	0	0	1	3	3	1	0	0	0	1	4	6	4	1	0	0	1	5	10	10	5	1	<table><tr><td>1</td><td>3</td><td>3</td><td>1</td><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>3</td><td>3</td><td>1</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>3</td><td>4</td><td>3</td><td>1</td><td>0</td></tr><tr><td>0</td><td>1</td><td>3</td><td>4</td><td>4</td><td>3</td><td>1</td></tr></table>	1	3	3	1	0	0	0	0	1	3	3	1	0	0	0	1	3	4	3	1	0	0	1	3	4	4	3	1	<table><tr><td>1</td><td>3</td><td>2</td><td>0</td><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>0</td><td>1</td><td>1</td><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>0</td><td>0</td><td>2</td><td>2</td><td>0</td><td>0</td></tr><tr><td>0</td><td>0</td><td>2</td><td>3</td><td>3</td><td>3</td><td>1</td></tr></table>	1	3	2	0	0	0	0	0	0	1	1	0	0	0	0	0	0	2	2	0	0	0	0	2	3	3	3	1
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(11) $\mathfrak{g} = (0, 0, 0, 12, 14, 15 + 23 + 24)$:

E_0							E_1							$E_2 = E_\infty$							
1	3	3	1	0	0	0	1	3	3	1	0	0	0	1	3	2	0	0	0	0	
0	1	3	3	1	0	0	0	1	3	3	1	0	0	0	0	1	1	0	0	0	
0	1	4	6	4	1	0	0	0	1	3	4	3	1	0	0	0	0	2	2	0	0
0	1	5	10	10	5	1	0	0	1	3	4	4	3	1	0	0	2	3	3	3	1

(12) $\mathfrak{g} = (0, 0, 0, 12, 14, 15 + 24)$: $\mathfrak{g} = \mathbb{R}e_3 \oplus \mathfrak{h}$ where \mathfrak{h} is the Lie algebra (1) of dimension 5.

E_0	E_1	$E_2 = E_\infty$																																																																																				
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(13) $\mathfrak{g} = (0, 0, 0, 12, 14, 15)$: $\mathfrak{g} = \mathbb{R}e_3 \oplus \mathfrak{h}$ where \mathfrak{h} is the Lie algebra (2) of dimension 5.

$$(14) \mathfrak{g} = (0, 0, 0, 12, 13, 14 + 35):$$

E_0

1	3	3	1	0	0	0
0	1	3	3	1	0	0
0	1	4	6	4	1	0
0	1	5	10	10	5	1

E_1

1	3	3	1	0	0	0
0	1	3	3	1	0	0
0	1	3	4	3	1	0
0	1	3	4	4	3	1

$E_2 = E_\infty$

1	3	2	0	0	0	0
0	0	1	1	0	0	0
0	0	0	2	2	0	0
0	0	2	3	3	3	1

$$(15) \mathfrak{g} = (0, 0, 0, 12, 23, 14 + 35):$$

E_0

1	3	3	1	0	0	0
0	2	7	9	5	1	0
0	1	5	10	10	5	1

E_1

1	3	3	1	0	0	0
0	2	6	6	3	1	0
0	1	3	6	6	3	1

$E_2 = E_\infty$

1	3	1	0	0	0	0
0	0	4	3	0	0	0
0	0	0	3	5	3	1

$$(16) \mathfrak{g} = (0, 0, 0, 12, 23, 14 - 35):$$

E_0

1	3	3	1	0	0	0
0	2	7	9	5	1	0
0	1	5	10	10	5	1

E_1

1	3	3	1	0	0	0
0	2	6	6	3	1	0
0	1	3	6	6	3	1

$E_2 = E_\infty$

1	3	1	0	0	0	0
0	0	4	3	0	0	0
0	0	0	3	5	3	1

$$(17) \mathfrak{g} = (0, 0, 0, 12, 14, 24): \mathfrak{g} = \mathbb{R}e_3 \oplus \mathfrak{h} \text{ where } \mathfrak{h} \text{ is the Lie algebra (3) of dimension 5.}$$

E_0

1	3	3	1	0	0	0
0	2	7	9	5	1	0
0	1	5	10	10	5	1

E_1

1	3	3	1	0	0	0
0	2	6	6	3	1	0
0	1	3	6	6	3	1

$E_2 = E_\infty$

1	3	1	0	0	0	0
0	0	4	3	0	0	0
0	0	0	3	5	3	1

$$(18) \mathfrak{g} = (0, 0, 0, 12, 13 - 24, 14 + 23):$$

E_0

1	3	3	1	0	0	0
0	1	3	3	1	0	0
0	2	9	16	14	6	1

E_1

1	3	3	1	0	0	0
0	1	3	3	1	0	0
0	2	6	7	5	3	1

$E_2 = E_\infty$

1	3	2	0	0	0	0
0	0	0	0	0	0	0
0	0	3	6	5	3	1

$$(19) \mathfrak{g} = (0, 0, 0, 12, 14, 13 - 24):$$

E_0

1	3	3	1	0	0	0
0	1	3	3	1	0	0
0	2	9	16	14	6	1

E_1

1	3	3	1	0	0	0
0	1	3	3	1	0	0
0	2	6	7	5	3	1

$E_2 = E_\infty$

1	3	2	0	0	0	0
0	0	0	0	0	0	0
0	0	3	6	5	3	1

$$(20) \mathfrak{g} = (0, 0, 0, 12, 13 + 14, 24):$$

E_0

1	3	3	1	0	0	0
0	1	3	3	1	0	0
0	2	9	16	14	6	1

E_1

1	3	3	1	0	0	0
0	1	3	3	1	0	0
0	2	6	7	5	3	1

$E_2 = E_\infty$

1	3	2	0	0	0	0
0	0	0	0	0	0	0
0	0	3	6	5	3	1

$$(21) \mathfrak{g} = (0, 0, 0, 12, 13, 14 + 23):$$

E_0

1	3	3	1	0	0	0
0	1	3	3	1	0	0
0	2	9	16	14	6	1

E_1

1	3	3	1	0	0	0
0	1	3	3	1	0	0
0	2	6	7	5	3	1

$E_2 = E_\infty$

1	3	2	0	0	0	0
0	0	0	0	0	0	0
0	0	3	6	5	3	1

$$(22) \mathfrak{g} = (0, 0, 0, 12, 13, 24):$$

E_0

1	3	3	1	0	0	0
0	2	7	9	5	1	0
0	1	5	10	10	5	1

E_1

1	3	3	1	0	0	0
0	2	6	6	3	1	0
0	1	3	6	6	3	1

$E_2 = E_\infty$

1	3	1	0	0	0	0
0	0	4	4	1	0	0
0	0	1	4	5	3	1

E_0	E_1	$E_2 = E_\infty$																																																															
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(23) $\mathfrak{g} = (0, 0, 0, 12, 13, 14)$:

E_0	E_1	$E_2 = E_\infty$																																																															
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(24) $\mathfrak{g} = (0, 0, 0, 12, 13, 23)$:

E_0	E_1	$E_2 = E_\infty$																																										
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(25) $\mathfrak{g} = (0, 0, 0, 0, 12, 15 + 34)$:

E_0	E_1	E_2	$E_3 = E_\infty$																																																																																				
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(26) $\mathfrak{g} = (0, 0, 0, 0, 12, 15)$: $\mathfrak{g} = \mathbb{R}e_3 \oplus \mathbb{R}e_4 \oplus \mathfrak{h}$ where \mathfrak{h} is the 3-step nilpotent Lie algebra of dimension 4.

E_0	E_1	$E_2 = E_\infty$																																																															
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0	0	1	4	6	4	1																																																											

(27) $\mathfrak{g} = (0, 0, 0, 0, 12, 14 + 25)$:

E_0	E_1	$E_2 = E_\infty$																																																															
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0	0	1	4	6	4	1																																																											

(28) $\mathfrak{g} = (0, 0, 0, 0, 13 + 42, 14 + 23)$:

E_0	E_1	$E_2 = E_\infty$																																										
<table><tr><td>1</td><td>4</td><td>6</td><td>4</td><td>1</td><td>0</td><td>0</td></tr><tr><td>0</td><td>2</td><td>9</td><td>16</td><td>14</td><td>6</td><td>1</td></tr></table>	1	4	6	4	1	0	0	0	2	9	16	14	6	1	<table><tr><td>1</td><td>4</td><td>6</td><td>4</td><td>1</td><td>0</td><td>0</td></tr><tr><td>0</td><td>2</td><td>8</td><td>11</td><td>8</td><td>4</td><td>1</td></tr></table>	1	4	6	4	1	0	0	0	2	8	11	8	4	1	<table><tr><td>1</td><td>4</td><td>4</td><td>0</td><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>0</td><td>4</td><td>10</td><td>8</td><td>4</td><td>1</td></tr></table>	1	4	4	0	0	0	0	0	0	4	10	8	4	1
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(29) $\mathfrak{g} = (0, 0, 0, 0, 12, 14 + 23)$:

E_0	E_1	$E_2 = E_\infty$																																										
<table><tr><td>1</td><td>4</td><td>6</td><td>4</td><td>1</td><td>0</td><td>0</td></tr><tr><td>0</td><td>2</td><td>9</td><td>16</td><td>14</td><td>6</td><td>1</td></tr></table>	1	4	6	4	1	0	0	0	2	9	16	14	6	1	<table><tr><td>1</td><td>4</td><td>6</td><td>4</td><td>1</td><td>0</td><td>0</td></tr><tr><td>0</td><td>2</td><td>8</td><td>11</td><td>8</td><td>4</td><td>1</td></tr></table>	1	4	6	4	1	0	0	0	2	8	11	8	4	1	<table><tr><td>1</td><td>4</td><td>4</td><td>0</td><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>0</td><td>4</td><td>10</td><td>8</td><td>4</td><td>1</td></tr></table>	1	4	4	0	0	0	0	0	0	4	10	8	4	1
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1	4	4	0	0	0	0																																						
0	0	4	10	8	4	1																																						

(30) $\mathfrak{g} = (0, 0, 0, 0, 12, 34)$:

E_0	E_1	$E_2 = E_\infty$																																										
<table><tr><td>1</td><td>4</td><td>6</td><td>4</td><td>1</td><td>0</td><td>0</td></tr><tr><td>0</td><td>2</td><td>9</td><td>16</td><td>14</td><td>6</td><td>1</td></tr></table>	1	4	6	4	1	0	0	0	2	9	16	14	6	1	<table><tr><td>1</td><td>4</td><td>6</td><td>4</td><td>1</td><td>0</td><td>0</td></tr><tr><td>0</td><td>2</td><td>8</td><td>11</td><td>8</td><td>4</td><td>1</td></tr></table>	1	4	6	4	1	0	0	0	2	8	11	8	4	1	<table><tr><td>1</td><td>4</td><td>4</td><td>0</td><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>0</td><td>4</td><td>10</td><td>8</td><td>4</td><td>1</td></tr></table>	1	4	4	0	0	0	0	0	0	4	10	8	4	1
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0	2	8	11	8	4	1																																						
1	4	4	0	0	0	0																																						
0	0	4	10	8	4	1																																						

(31) $\mathfrak{g} = (0, 0, 0, 0, 12, 13)$: $\mathfrak{g} = \mathbb{R}e_4 \oplus \mathfrak{h}$ where \mathfrak{h} is the Lie algebra (6) of dimension 5.

E_0							E_1							$E_2 = E_\infty$						
1	4	6	4	1	0	0	1	4	6	4	1	0	0	1	4	4	1	0	0	0
0	2	9	16	14	6	1	0	2	8	12	9	4	1	0	0	5	11	9	4	1

(32) $\mathfrak{g} = (0, 0, 0, 0, 0, 12 + 34)$: $\mathfrak{g} = \mathbb{R}e_5 \oplus \mathfrak{h}$ where \mathfrak{h} is the Lie algebra (8) of dimension 5.

E_0	E_1	$E_2 = E_\infty$																																										
<table><tr><td>1</td><td>5</td><td>10</td><td>10</td><td>5</td><td>1</td><td>0</td></tr><tr><td>0</td><td>1</td><td>5</td><td>10</td><td>10</td><td>5</td><td>1</td></tr></table>	1	5	10	10	5	1	0	0	1	5	10	10	5	1	<table><tr><td>1</td><td>5</td><td>10</td><td>10</td><td>5</td><td>1</td><td>0</td></tr><tr><td>0</td><td>1</td><td>5</td><td>10</td><td>10</td><td>5</td><td>1</td></tr></table>	1	5	10	10	5	1	0	0	1	5	10	10	5	1	<table><tr><td>1</td><td>5</td><td>9</td><td>5</td><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>0</td><td>0</td><td>5</td><td>9</td><td>5</td><td>1</td></tr></table>	1	5	9	5	0	0	0	0	0	0	5	9	5	1
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1	5	9	5	0	0	0																																						
0	0	0	5	9	5	1																																						

(33) $\mathfrak{g} = (0, 0, 0, 0, 0, 12)$: $\mathfrak{g} = \mathbb{R}e_3 \oplus \mathbb{R}e_4 \oplus \mathfrak{h}$ where \mathfrak{h} is the nilpotent Lie algebra of dimension 3.

E_0	E_1	$E_2 = E_\infty$																																										
<table><tr><td>1</td><td>5</td><td>10</td><td>10</td><td>5</td><td>1</td><td>0</td></tr><tr><td>0</td><td>1</td><td>5</td><td>10</td><td>10</td><td>5</td><td>1</td></tr></table>	1	5	10	10	5	1	0	0	1	5	10	10	5	1	<table><tr><td>1</td><td>5</td><td>10</td><td>10</td><td>5</td><td>1</td><td>0</td></tr><tr><td>0</td><td>1</td><td>5</td><td>10</td><td>10</td><td>5</td><td>1</td></tr></table>	1	5	10	10	5	1	0	0	1	5	10	10	5	1	<table><tr><td>1</td><td>5</td><td>9</td><td>7</td><td>2</td><td>0</td><td>0</td></tr><tr><td>0</td><td>0</td><td>2</td><td>7</td><td>9</td><td>5</td><td>1</td></tr></table>	1	5	9	7	2	0	0	0	0	2	7	9	5	1
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From the simple inspection of the tables above we can see that in dimension five there are 6 different configurations of tables that correspond to 8 isomorphisms classes of nilpotent Lie algebras in that dimension. Meanwhile, there are 15 different configuration of tables that correspond to 33 isomorphisms classes of nilpotent Lie algebras in dimension six. This allow us to state the following result.

Proposition 5.1. *There are non-isomorphic nilpotent Lie algebras with the same intermediate cohomology diagrams.*

Remark. Notice that the the diagrams of the Lie algebras of dimension six (16) and (17) have different intermediate cohomology configurations but the same Betti numbers.

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